Analytic Solution for CVA of a Collateralised Call Option

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CVA is a hard problem!

European equity options can be priced and risk-managed with an analytic formula. But if you want the associated CVA:

- you need a credit default intensity model;
- option pricing becomes an integral over the conditional price of a compound option;
- possible equity-intensity correlation requires explicit stochastic modelling for *both* processes;
- equity may jump at default so survival/default-contingent equity models are needed;
- collateral may be updated dynamically so an option becomes a portfolio of forward-starting options;
- possible netting for a counterparty prevents CVA being defined at a per-instrument level.

Analytic Pricing versus Monte Carlo?

- Normally use Monte Carlo simulation for such problems.
- Even then ad hoc assumptions often needed about future *exposure* distribution.

We will show how instead the Black-Scholes framework can be extended $\rightarrow$ compute CVA analytically.

Overview of Presentation

- Stochastic modelling of credit intensity and equity price
- Derivation of governing PDE
- Asymptotic representation of solution to PDE
- Approximate calculation of CVA for call/put options (incorporating jump and correlation risk)
- Extension for dynamic CVA update
- Extension to portfolio of options (with single counterparty)

Modelling assumptions

- Rates are deterministic.
- Credit intensity follows a Black-Karasinski short rate model [BK91]
- Equity price follows a lognormal diffusion process with an assumed jump of relative size $k < 0$ in the event of a default of the option-writing counterparty.
- Equity and credit diffusions are assumed correlated (typically negatively).
- Collateral follows a predetermined schedule.

Credit model

Following [Turfus16a], let credit intensity λ_t be defined by

$$\lambda_t = (\overline{\lambda}(t) + \lambda^*(t))\mathcal{E}(y_t), \tag{1}$$

where the auxiliary process y_t is governed by

$$dy_t = -\alpha y_t dt + \sigma_y(t)\, dW_t^1, \tag{2}$$

$\overline{\lambda}(t)$ is defined implicitly by

$$E\left[e^{-\int_0^t \lambda_s\, ds}\right] = e^{-\int_0^t \overline{\lambda}(s)\, ds} \tag{3}$$

and $\lambda^*(t)$ is to be determined by calibration. The survival probabilities defined in Eq. (3) are assumed known from market data.

Equity model

For the equity process we propose, extending the approach of [Turfus16b]:

$$\frac{dS_t}{S_t} = \begin{cases} (\bar{r}(t) - q(t) - k\lambda_t)\,dt + \sigma_1(t)\,dW_t^2 + k dn_t & \text{if } t \le \tau, \\ (\bar{r}(t) - q(t) + \sigma_2(t)\,dW_t^2 & \text{if } t > \tau, \end{cases} \tag{4}$$

with τ the counterparty default time. We expect calibration to option prices will result in $\sigma_1(t) < \sigma_2(t)$.
Further suppose

$$\text{corr}(W_t^1, W_t^2) = \rho_{\lambda S}, \tag{5}$$

typically negative.

Variances and Discount Factors

Define term variances and term covariance:

$$
\begin{aligned}
I_1(t_1, t_2) &:= \int_{t_1}^{t_2} \sigma_1^2(u)du. \\
I_2(t_1, v, t_2) &:= \int_{t_1}^{v} \sigma_1^2(u)du + \int_{v}^{t_2} \sigma_2^2(u)du. \\
I_y(t_1, t_2) &:= \int_{t_1}^{t_2} e^{-2\alpha(t_2-u)}\sigma_y^2(u)du \\
I_\rho(t_1, t_2) &:= \rho_{\lambda S} \int_{t_1}^{t_2} e^{-a(t_2-u)}\sigma_y(u)\sigma_1(u)du,
\end{aligned}
$$

Let $D(t_1, t_2)$ be the discount factor and

$$B(t_1, t_2) = D(t_1, t_2)e^{-\int_{t_1}^{t_2} \overline{\lambda}(s)\, ds}$$

the corresponding risky discount factor.

Change of Equity Variable

Express the equity price S_t in terms of a new auxiliary process x_t, conditional on default at time $\tau = v$ through

$$S_t = \begin{cases} F_1(t)e^{x_t - \frac{1}{2}I_1(0,t)} & \text{if } t < v, \\ (1+k)F_2(v,t)e^{x_t - \frac{1}{2}I_2(0,v,t)} & \text{if } t \geq v, \end{cases} \tag{6}$$

where

$$F_1(t) \quad := \quad S_0 e^{\int_0^t (\bar{r}(s) - q(s) - k\bar{\lambda}(s))ds}, \tag{7}$$

$$F_2(v,t) \quad := \quad F_1(v)e^{\int_v^t (\bar{r}(s) - q(s))ds}, \tag{8}$$

Also define

$$M_1(x,t) \quad := \quad e^{x - \frac{1}{2}I_1(0,t)}F_1(t),$$
$$M_2(x,v,t) \quad := \quad e^{x - \frac{1}{2}I_2(0,v,t)}F_2(v,t).$$

CVA Payoff

The price of a call option with strike K and maturity T_m will be given by the standard Black formula:

$$f_m(x,t) = D(t,T_m)\left((1+k)M_2(x,t,T_m)N(d_1(x,t,T_m)) - KN(d_2(x,t,T_m))\right), \tag{9}$$

where

$$\begin{aligned} d_2(x,t,u) &:= \frac{\ln((1+k)M_2(x,t,u)) - \ln K}{\sqrt{I_2(t,u)}}, \\ d_1(x,t,u) &:= d_2(x,t,u) + \sqrt{I_2(t,u)}. \end{aligned}$$

Thus the payoff function for default at time $\tau = v$ with $x_v = \xi$ is given by

$$P_m(\xi,v) = (1-R)\max\left\{f_m(\xi,v) - C(v), 0\right\}. \tag{10}$$

with $C(t)$ the collateral at time t and R the counterparty recovery rate (henceforth assumed for notational convenience to be zero).

PDE Representation

Suppose CVA(t) can be expressed as $h(x_t, y_t, t)$. We want CVA(0) $\equiv h(0,0,0)$. Applying the well-known Feynman-Kac method we obtain:

$$\mathcal{L}[h(x,y,t)] = -\left(\overline{\lambda}(t) + \Delta\lambda(y,t)\right) P_m(x,t) + \Delta\lambda(y,t)\left(h + k\frac{\partial h}{\partial x}\right) \quad (11)$$

for $t \in D_m$ with final condition that $h(x,y,T_m) = 0$, where $\mathcal{L}[.]$ is a forced diffusion operator given by

$$\mathcal{L}[\cdot] := \frac{\partial}{\partial t} - \alpha y\frac{\partial}{\partial y} + \tfrac{1}{2}\sigma_1^2(t)\frac{\partial^2}{\partial x^2} + \rho_{\lambda S}\sigma_1(t)\sigma_y(t)\frac{\partial^2}{\partial x \partial y} + \tfrac{1}{2}\sigma_y^2(t)\frac{\partial^2}{\partial y^2} - (\bar{r}(t) + \overline{\lambda}(t)) \quad (12)$$

and

$$\Delta\lambda(y,t) := (\overline{\lambda}(t) + \lambda^*(t))\mathcal{E}_y(y_t) - \overline{\lambda}(t)$$

with $\mathcal{E}_y(y_t) := \mathcal{E}(y_t)|_{y_t=y}$. We observe that $L[\cdot]$ thus defined has a well-known Green's function solution. . .

Asymptotic Scaling

Following [Turfus16a], we suppose the credit intensity to be small and define an asymptotically small parameter ϵ by

$$\epsilon := \frac{1}{\alpha T_m} \int_0^{T_m} \overline{\lambda}(t) dt.$$

We then define an $O(1)$ scaled forward intensity $\tilde{\lambda}(t)$ by

$$\tilde{\lambda}(t) := \epsilon^{-1} \overline{\lambda}(t) \tag{13}$$

and further define

$$\Delta\tilde{\lambda}(y, t) = \epsilon^{-1} \Delta\lambda(y, t) \tag{14}$$

whence we propose

$$h(x, y, t) = \epsilon h_1(x, y, t) + \epsilon^2 h_2(x, y, t) + O(\epsilon^3). \tag{15}$$

Perturbation Analysis

At first order we must solve

$$\mathcal{L}[h_1(x,y,t)] = -\tilde{\lambda}(t)\mathcal{E}_y(y_t)P_m(x,t)$$

for $t \in D_m$ with final condition that $h_1(x, T_m) = 0$. This can be achieved by means of the following readily obtainable Green's function for the diffusion operator $\mathcal{L}[\cdot]$:

$$G(x,y,t;\xi,\eta,v) = B(t,v)\frac{\partial^2}{\partial\xi\partial\eta}N_2\left(\frac{x-\xi}{\sqrt{I_1(t,v)}}, \frac{ye^{-\alpha(v-t)}-\eta}{\sqrt{I_y(t,v)}}; \rho(t,v)\right) \quad (16)$$

for $0 \le t \le v$, where $N_2(x_1, x_2; \rho)$ is a standard bivariate cumulative Gaussian distribution with correlation ρ and

$$\rho(t,v) := \frac{I_\rho(t,v)}{\sqrt{I_1(t,v)I_y(t,v)}}.$$

Theorem: Call Option CVA

Applying the Green's function, setting $x = y = t = 0$ and reverting to unscaled notation we obtain:

Theorem

The CVA for a European call option on an equity underlying can be estimated under our modelling assumptions as follows:

$$
\begin{aligned}
CVA(0) = & \int_0^{T_m} B(0,v)\overline{\lambda}(v)(1+k)D(v,T_m)F_2(v,T_m)e^{I_\rho(0,v)}\psi_1^+(v,T_m)\,dv \\
& - \int_0^{T_m} B(0,v)\overline{\lambda}(v)\left(D(v,T_m)K\psi_2^+(v,T_m) + C(v)N(a_2(I_\rho(0,v),0,v))\right)dv \\
& + O(\epsilon^2)
\end{aligned}
\tag{17}
$$

Proof. The proof is by construction, by analogy with the calculation of [Geske79]. □

Notation

In stating the above theorem we have defined

$$
\begin{aligned}
\psi_i^{\pm}(v,w) &:= N_2(\pm a_i(I_\rho(0,v),0,v), \pm b_i(I_\rho(0,v),0,v,w), R(0,v,w)),\\
\xi^*(v) &:= \sup\{\xi \mid P_m(\xi,v)=0\},\\
a_2(x,t,v) &:= \frac{x-\xi^*(v)}{\sqrt{I_1(t,v)}},\\
a_1(x,t,v) &:= a_2(x,t,v)+\sqrt{I_1(t,v)},\\
b_2(x,t,v,w) &:= \frac{\ln((1+k)M_2(x,v,w))-\ln K}{\sqrt{I_2(t,v,w)}},\\
b_1(x,t,v,w) &:= b_2(x,t,v,w)+\sqrt{I_2(t,v,w)},\\
R(t,v,w) &:= \sqrt{\frac{I_1(t,v)}{I_2(t,v,w)}}, \quad v<w.
\end{aligned}
$$

Note that in terms of our previous notation

$$b_i(x,v,v,w) \equiv d_i(x,v,w), \quad i=1,2$$

Theorem: Put Option CVA

Reusing the previous notation but interpreting the option price as

$$f_m(x,t) = D(t,T_m)\left(KN(-d_2(x,t,T_m)) - (1+k)M_2(x,t,T_m)N(-d_1(x,t,T_m))\right) \tag{18}$$

for the put, and $\xi^*(v)$ as inf rather than sup, we have

Theorem

The CVA on a European put option on an equity underlying can be estimated under our modelling assumptions as follows:

$$\begin{aligned} CVA(0) = &\int_0^{T_m} B(0,v)\overline{\lambda}(v)\left(D(v,T_m)K\psi_2^-(v,T_m) - C(v)N(-a_2(I_\rho(0,v),0,v))\right)dv \\ &- \int_0^{T_m} B(0,v)\overline{\lambda}(v)(1+k)D(v,T_m)F_2(v,T_m)e^{I_\rho(0,v)}\psi_1^-(v,T_m)\,dv + O(\epsilon^2). \end{aligned} \tag{19}$$

Proof. The proof is analogous to that for the call option. □

Dynamically Updated Collateral

- Suppose at scheduled dates t_j, $j = 0, 1, \ldots, n-1$ the collateral is updated to reflect the current exposure level associated with the option, viz. its PV.
- Each exposure period $[t_j, t_{j+1})$ can be considered separately, with the compound option taken to be forward-starting at t_j with strike given by the value at time t_j of the option, conditional on survival of the option writer.
- Suppose the PV of the option for collateral calculation purposes at $t = t_j$ can be calculated assuming the forward evolution of S_t to be governed by the *second* line of Eq. (4). (This should be a reasonable assumption if the equity model is well calibrated.)
- Thus we propose

$$S_t = F_2(t_j, t)e^{x_t - \frac{1}{2}I_2(0,t_j,t)}, \quad t \geq t_j \geq 0$$

New CVA Payoff

We infer the market value of the option as of time t_j conditional on $x_{t_j} = x$ and $\tau > t_j$ is

$$C_j(x) = D(t_j, T_m)(M_2(x, t_j, T_m)N(d_1^*(x, t_j, T_m)) - KN(d_2^*(x, t_j, T_m))), \tag{20}$$

where

$$\begin{aligned} d_2^*(x, t, u) &:= \frac{\ln M_2(x, t, u) - \ln K}{\sqrt{I_2(t, u)}}, \\ d_1^*(x, t, u) &:= d_2^*(x, t, u) + \sqrt{I_2(t, u)} \end{aligned}$$

The payoff function we must now consider conditional on $x_{t_j} = x_j$ for $j = 0, 1, \ldots, n-1$ becomes

$$P_m^*(x, t) := (1 - R) \max \{f_m(x, t) - C_j(x_j), 0\} (H(t - t_j) - H(t - t_{j+1})),$$

Theorem: Call Option CVA – Collateral Update

Theorem

The CVA on a European call option on an equity underlying associated with default in a forward time period $[t_j, t_{j+1})$ where the collateral is reset at t_j can be estimated under our modelling assumptions as follows:

$$
\begin{aligned}
CVA^{(j)}(0) \approx \int_{t_j}^{t_{j+1}} & B(0,v)D(v,T_m)\overline{\lambda}(v)(1+k)F_2(v,T_m)e^{I_\rho(0,v)}\psi_1^{(j)+}(I_\rho(0,v),v,T_m)\,dv \\
& - K\int_{t_j}^{t_{j+1}} B(0,v)D(v,T_m)\overline{\lambda}(v)\psi_2^{(j)+}(I_\rho(0,v),v,T_m)\,dv \\
& - \int_{t_j}^{t_{j+1}} \overline{\lambda}(v)B(0,v)D(v,T_m)N\left(a_2^{(j)}(I_\rho(t_j,v),t_j,v)\right)dv \\
& \left(e^{I_\rho(0,t_j)}F_2(t_j,T_m)N(b_1^{(j)}(I_\rho(0,t_j),0,T_m)) - KN(b_2^{(j)}(I_\rho(0,t_j),0,T_m))\right).
\end{aligned}
$$

Notation

In stating the above theorem we have defined

$$\begin{aligned}
\psi_i^{(j)\pm}(v,w) &:= N_2(\pm a_i^{(j)}(I_\rho(0,v),0,v), \pm b_i^{(j)}(I_\rho(0,v),0,v,w), R(0,v,w)),\\
\xi_j^*(x,v) &:= \inf\{\xi \mid f_m(\xi,v) - C_j(x) > 0\},\\
\xi_j^*(v) &:= \xi_j^*(0,v),\\
a_2^{(j)}(x,t,v) &:= \frac{x - \xi_j^*(v)}{\sqrt{I_1(t,v)}},\\
a_1^{(j)}(x,t,v) &:= a_2(x,t,v) + \sqrt{I_1(t,v)},\\
b_2^{(j)}(x,t,w) &:= \frac{\ln M_2(x,t_j,w) - \ln K}{\sqrt{I_2(t,t_j,w)}},\\
b_1^{(j)}(x,t,w) &:= b_2(x,t,w) + \sqrt{I_2(t,t_j,w)}.
\end{aligned}$$

We have also used the approximation that $\xi_j^*(x,v) \approx x + \xi_j^*(v)$ to facilitate the analysis.

A similar result can be derived for the put option. For more details, see [Turfus17].

Further Developments

- The approach described is readily applicable to CVA calculation for FX and commodity options. It should also be extensible to other European-style options such as digitals.
- Work is ongoing to examine the impact of the correlation and jump risk on the CVA in practical contexts and to ascertain the reliability of the approximation methods used.
- More details of the above calculation are available in the companion working paper, see [Turfus17].

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